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Simultaneous multiple capture in a simple pursuit problem☆

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ABSTRACT

The necessary and sufficient conditions for simultaneous multiple capture in a simple group pursuit problem with different opportunities for the participants are obtained.

The problem of simple group pursuit with different opportunities was considered for the first time by Pshenichnyi and the necessary and sufficient conditions for capture were obtained.[1] The necessary and sufficient conditions for multiple capture have been presented[2] in the case of a problem with simple motions and equal opportunities. The sufficient conditions for multiple capture have been obtained[3] in Pontryagin's example with equal opportunities. Multiple capture occurs if a given number of pursuers catch an evader but the times at which the evader is captured can be different. In the simultaneous multiple capture problem, the capture times are identical.

1. Formulation of the problem

A differential game Γ involving $n+1$ persons is considered in the space R^ν ($\nu \geq 2$). There are n pursuers $P_1, P_2, \ldots, P_n$ and an evader E with the laws of motion and initial conditions (when $t=t_0$)

$$\dot{x}_i = u_i, \quad u_i \in V, \quad x_i(t_0) = X_i^0, \quad i \in I, \quad \dot{y} = \upsilon, \quad \upsilon \in V, \quad y(t_0) = Y^0 \tag{1.1}$$

and $X_i^0 \neq Y^0$ for all $i \in I = \{1,2,\ldots,n\}$. Here, x_i, $y \in R^\nu$, V is a strictly convex compactum in R^ν with a smooth boundary such that $V \neq \emptyset$. For each $k=1,2,\ldots,n$, we define a set

$$\Omega(k) = \{(i_1, i_2, \ldots, i_k)\colon i_1, i_2, \ldots, t_k \in I \ \text{Pait wise different}\}$$

We shall call controls from the class of Lebesgue-measurable functions in $[t_0,\infty)$ with values from the set V permissible controls. A mapping U_i which matches a permissible control $u_i(t)$ with the instant t, the initial conditions X_i^0, Y^0 and an arbitrary permissible prehistory of the evader control $\upsilon(s)$, $t_0 \leq s \leq t$, that is,

$$u_i(t) = U_i(t, X_i^0, Y^0, \upsilon(s), t_0 \leq s \leq t)$$

is called a quasistrategy of a pursuer P_i.

It is assumed here that a "physical feasibility" condition must be satisfied: if v^1, v^2 are two permissible controls of the evader E and $v^1(t) = v^2(t)$ for almost all $t \in [t_0,\infty)$, then the functions u^1 and u^2 corresponding to them in the mapping U_i are also equal for almost all $t \in [t_0,\infty)$.

Definition 1. A simultaneous m-tuple capture $n \geq m \geq 1$ is possible in the game Γ if an instant $T_0 = T_0(X_i^0, Y^0)$ and quasistrategies U_i of the pursuers P_i exist such that, for any permissible control $v(t)$ of the evader E, an instant $\tau \in [t_0,T_0]$ and a set $\Lambda \in \Omega(m)$ are found for which the condition

$$x_\alpha(\tau) = y(\tau), \quad x_\alpha(s) \neq y(s) \quad \text{for all} \quad s \in [t_0, \tau), \quad \alpha \in \Lambda$$

is satisfied.

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2. Solution of the problem

Introducing the replacement $z_i = x_i - y$, we rewrite relations (1.1) in the form

$$\dot{z}_i = u_i - \upsilon, \quad u_i, \upsilon \in V, \quad z_i(t_0) = Z_i^0 = X_i^0 - Y^0 \neq 0, \quad i = I$$

We now introduce the notation

$$e_i^0 = \frac{Z_i^0}{\|Z_i^0\|}, \quad \lambda_i(\upsilon) = \sup\{\lambda \geq 0: (\upsilon - \lambda e_i^0) \in V\}, \quad \delta = \min_{\upsilon \in V} \max_{\Lambda \in \Omega(m)} \min_{\alpha \in \Lambda} \lambda_\alpha(\upsilon)$$

Condition 1. $0 \in \operatorname{Intco}\{Z_k^0,\ k \in K\}$ for all sets $K \in \Omega(n-m+1)$.

Lemma 1. *Suppose Condition 1 is satisfied. Then,* $\delta > 0$.

Proof. We assume that, contrary to the assertion, $\delta = 0$. An element of $w \in V$ is then found such that an element $\alpha \in \Lambda$ for which $\lambda_\alpha(w) = 0$ exists in any set $\Lambda \in \Omega(m)$. We now construct the set $Q = \{q_1, q_2, \ldots, q_{n-m+1}\} \in \Omega(n-m+1)$ according to the following rule. We select an element $q_1 \in L_1 = \{1, 2, \ldots, m\} \in \Omega(m)$ from the condition $\lambda_{q_1}(w) = 0$ and then an element $q_2 \in L_2 = (L_1 \cup \{m+1\}) \backslash \{q_1\} \in \Omega(m)$ such that $\lambda_{q_2}(w) = 0$ and then an element $q_3 \in L_3 = (L_2 \cup \{m+2\}) \backslash \{q_2\} \in \Omega(m)$ which satisfies the equality $\lambda_{q_3}(w) = 0$, and so on. In the last step we construct the set

$$L_{n-m+1} = (L_{n-m} \cup \{n\}) \backslash \{q_{n-m}\} \in \Omega(m)$$

and choose an element $q_{n-m+1} \in L_{n-m+1}$ according to the condition $\lambda_{q_{n-m+1}}(w) = 0$. By construction, the equality

$$\min_{\upsilon \in V} \max_{q \in Q} \lambda_q(\upsilon) = 0$$

holds for the set $Q \in \Omega(n-m+1)$ whence it follows that $0 \notin \operatorname{Intco}\{Z_q^0, q \in Q\}$ and Condition 1 is not satisfied. The resulting contradiction proves that $\delta > 0$.

Theorem 1. *In the game* Γ, *a simultaneous n-tuple capture is possible if and only if Condition 1 is satisfied.*

Proof *(Sufficiency).* Suppose Condition 1 is satisfied. We construct the permissible controls $u_i(t)$ of the pursuers P_i, which ensure a simultaneous n-tuple capture in the case of an arbitrary permissible control $\upsilon(t)$ of the evader E. We introduce the notation

$$T_i(t) = \begin{cases} \dfrac{\|z_i(t)\|}{\lambda_i(\upsilon(t))}, & \text{if } \lambda_i(\upsilon(t)) > 0 \\ \infty, & \text{if } \lambda_i(\upsilon(t)) = 0 \end{cases}$$

$$T(t, \Lambda) = \max_{\alpha \in \Lambda} T_\alpha(t), \quad \Lambda(t) = \arg \min_{\Lambda \in \Omega(m)} T(t, \Lambda)$$

$$h_i(t) = \begin{cases} \dfrac{T_i(t)}{T(t, \Lambda(t))}, & \text{if } i \in \Lambda(t) \text{ и } T(t, \Lambda(t)) > 0 \\ 1 & \text{otherwise} \end{cases} \tag{2.1}$$

We now specify the controls $u_i(t)$ of the pursuers P_i as follows:

$$u_i(t) = \upsilon(t) - h_i(t)\lambda_i(\upsilon(t))e_i^0 \quad \text{for all} \quad t \in [t_0, \infty)$$

Note that the controls $u_i(t)$ are permissible. Suppose

$$\tau = \min\{t > t_0: \min_{i \in I}\|z_i(t)\| = 0\}$$

The possibility of a simultaneous m-tuple capture is equivalent to the following assertions holding.

1°. An instant $T_0 = T_0(X_i^0, Y^0) < \infty$ exists such that $\tau \in [t_0, T_0]$.
2°. A set $\Lambda \in \Omega(m)$ is found such that $\|z_\alpha(\tau)\| = 0$ for all $\alpha \in \Lambda$, that is, not one but at least m of the n quantities $\|z_\alpha(\tau)\|$, $i \in I$ vanish at the instant τ.

We now prove Assertion 1°. By the Cauchy formula,

$$z_i(t) = Z_i^0 - e_i^0 H_i(t) = e_i^0(\|Z_i^0\| - H_i(t)), \quad H_i(t) = \int_{t_0}^{t} h_i(s)\lambda_i(\upsilon(s))ds \tag{2.2}$$

for all $t\in[t_0,\infty)$. For each $t\in[t_0,\tau)$ from (2.1) we have $h_i(t)=1$, $i\notin\Lambda(t)$ and, furthermore, a subscript $k\in\Lambda(t)$ exists such that $T_k(t)=T(t, \Lambda(t))$, that is, $h_k(t)=1$. This means that, of the n values of $h_i(t)$, $i\in I$, no less than $n-m+1$ are equal to unity. Consequently, an index $k\in arg \max\limits_{\Lambda\in\Omega(m)}\min\limits_{\alpha\in\Lambda}\lambda_\alpha$ exists for all $t\in[t_0,\tau)$ such that $h_k(t)=1$ and

$$\max_{i\in I}h_i(t)\lambda_i(\upsilon(t))\geq h_k(t)\lambda_k(\upsilon(t)) = \lambda_k(\upsilon(t))\geq\delta \tag{2.3}$$

By virtue of relations (2.2) and (2.3), we obtain that

$$\min_{i\in I}\|z_i(t)\| = \min_{i\in I}(\|Z_i^0\|-H_i(t))\leq\max_{i\in I}\|Z_i^0\|-\max_{i\in I}H_i(t)\leq$$
$$\leq\max_{i\in I}\|Z_i^0\|-\frac{1}{n}\sum_{i\in I}H_i(t)\leq\max_{i\in I}\|Z_i^0\|-\frac{\delta}{n}(t-t_0)$$

for all $t\in[t_0,\tau)$. Hence, no later than the instant

$$T_0 = t_0+\delta^{-1}n\max_{i\in I}\|Z_i^0\|$$

just one of the quantities $||z_i(t)||$ vanishes. Assertion 1° is proved.

We assume that Assertion 2° is untrue, which means that sets Q and R exist such that

$$Q\subset I,\quad 1\leq|Q|\leq m-1,\quad R = I\backslash Q,\quad |R|\geq n-m+1$$
$$\|z_q(\tau)\| = 0,\quad q\in Q,\quad \|z_r(\tau)\|>0,\ r\in R \tag{2.4}$$

Then, $\varepsilon_1>0$ and $c_1>0$ exist such that, for all $q\in Q$ and $r\in R$,

$$\frac{d}{dt}\|z_q(t)\| = -h_q(t)\lambda_q(\upsilon(t))<0 \quad\text{for almost all}\quad t\in[\tau-\varepsilon_1,\tau) \tag{2.5}$$

$$\|z_r(t)\|\geq c_1>0 \quad\text{for all}\quad t\in[t_0,\tau] \tag{2.6}$$

Next, we obtain from relations (2.1) and (2.4) that, for all $t\in[\tau-\varepsilon_1,\tau)$, a number $k\in R\cap\Lambda(t)$ is found and

$$T(t,\Lambda(t)) = \max_{\alpha\in\Lambda(t)}T_\alpha(t)\geq T_k(t) = \frac{\|z_k(t)\|}{\lambda_k(\upsilon(t))}\geq\frac{c_1}{2\mathrm{diam}(V)} = c_2>0 \tag{2.7}$$

since the inequalities (2.6) and $\lambda_1(v)\leq 2\mathrm{diam}(V)$ hold by virtue of the definition of the quantity λ_i and the fact that $||e_i^0||$ for all $i\in I$, $v\in V$, where $\mathrm{diam}(V)=\max\{||v||: v\in V\}$. Note that, by virtue of relation (2.5), the functions $||z_q(t)||$, $q\in Q$ are strictly decreasing in the segment $\tau-\varepsilon_1,\tau$, and therefore

$$\|z_q(s_1)\|\geq\|z_q(s_2)\| \quad\text{for all}\quad q\in Q,\quad s_1\leq s_2,\quad s_1,s_2\in[\tau-\varepsilon_1,\tau] \tag{2.8}$$

We now assume that $\varepsilon\in(0,\min\{\varepsilon_1,c_2\})$ and $k\in Q$ exist such that $k\notin\Lambda(t)$ for almost all $t\in[\tau-\varepsilon,\tau)$. Then, $h_k(t)=1$, and from relation (2.5) we obtain that

$$\lambda_k(\upsilon(t)) = -\frac{d}{dt}\|z_q(t)\| \quad\text{for almost all}\quad t\in[\tau-\varepsilon,\tau)$$

It then follows from relations (2.1) and (2.7) that

$$\frac{\|z_k(t)\|}{-\frac{d}{dt}\|z_k(t)\|} = \frac{\|z_k(t)\|}{\lambda_k(\upsilon(t))}\geq T(t,\Lambda(t))\geq c_2>0 \quad\text{for almost all}\quad t\in[\tau-\varepsilon,\tau)$$

whence

$$\frac{d}{dt}\|z_k(t)\|\geq -c_2^{-1}\|z_k(t)\| \quad\text{for almost all}\quad t\in[\tau-\varepsilon,\tau)$$

Taking account of the last inequality and relations (2.4) and (2.8), we have

$$0 = \|z_k(\tau)\| = \|z_k(\tau-\varepsilon)\|+\int_{\tau-\varepsilon}^{\tau}\left(\frac{d}{ds}\|z_k(s)\|\right)ds\geq$$
$$\geq\|z_k(\tau-\varepsilon)\|-\int_{\tau-\varepsilon}^{\tau}c_2^{-1}\|z_k(s)\|ds\geq\|z_k(\tau-\varepsilon)\|-\int_{\tau-\varepsilon}^{\tau}c_2^{-1}\|z_k(\tau-\varepsilon)\|ds =$$
$$= \|z_k(\tau-\varepsilon)\|(1-c_2^{-1}\varepsilon)>0$$

The assumption that the above mentioned ε and k exist is untrue, and this, in turn, means that a number $\varepsilon_2 \in (0, \min\{\varepsilon_1, c_2\})$ exists such that $Q \subset \Lambda(t)$ for almost all $t \in \{\tau - \varepsilon_2, \tau)$. Then, for all $q \in Q$,

$$h_q(t) = \frac{T_q(t)}{T(t, \Lambda(t))} \quad \text{for almost all} \quad t \in [\tau - \varepsilon_2, \tau) \tag{2.9}$$

We now select an arbitrary instant $t \in [\tau - \min\{\varepsilon_2, c_2/2, \tau\})$ and, from relations (2.2), (2.7), (2.8) and (2.9), we obtain that the following chain of equalities inequalities holds for all $q \in Q$

$$0 = \|z_q(\tau)\| = \left\| z_q(t) - \int_t^\tau h_q(s)\lambda_q(\upsilon(s))e_q^0 ds \right\| \geq$$

$$\geq \|z_q(t)\| - \int_t^\tau h_q(s)\lambda_q(\upsilon(s))ds = \|z_q(t)\| - \int_t^\tau \frac{T_q(s)}{T(s, \Lambda(s))}\lambda_q(\upsilon(s))ds \geq$$

$$\geq \|z_q(t)\| - \frac{1}{c_2}\int_t^\tau T_q(s)\lambda_q(\upsilon(s))ds = \|z_q(t)\| - \frac{1}{c_2}\int_t^\tau \|z_q(s)\| ds \geq$$

$$\geq \|z_q(t)\| - \frac{1}{c_2}\|z_q(t)\|(\tau - t) \geq \|z_q(t)\| - \frac{1}{c_2}\|z_q(t)\|\frac{c_2}{2} = \frac{\|z_q(t)\|}{2} = 0.$$

The resulting contradiction proves Assertion 2°.

Necessity. Suppose Condition 1 is not satisfied. This means that a set $Q \in \Omega(n - m + 1)$ exists such that $0 \notin \operatorname{Intco}\{Z_q^0, q \in Q\}$. It follows from the separability theorem that a unique vector p exists such that $\langle h,p \rangle \leq 0$ for all $h \in \operatorname{co}\{Z_q^0, q \in Q\}$, and therefore,

$$\langle Z_q^0, p \rangle \leq 0 \quad \text{for all} \quad q \in Q$$

We define the constant control of the evader E as follows:

$$\upsilon(t) = \upsilon_p \quad \text{for all} \quad t \in [t_0, \infty)$$

The vector $\upsilon_p \in \partial V$ is chosen from the condition $\langle u - \upsilon_p, p \rangle < 0$ for all $u \in V \backslash \{\upsilon_p\}$. Such a vector exists and it is unique since V is a strictly convex compactum in R^ν. Then, for all $q \in Q$ and $t > t_0$,

$$\langle z_q(t), p \rangle = \langle Z_q^0, p \rangle + \int_{t_0}^t \langle u_q(s) - \upsilon_p, p \rangle ds \leq 0$$

and the equality is only possible in the case when $u_q(s) = \upsilon_p$ almost everywhere in $[t_0, t]$ but, in this case, $z_q(t) = Z_q^0 \neq 0$. If $\langle z_q(t),p \rangle < 0$, then $z_q(t) \neq 0$. Consequently, $z_q(t) \neq 0$ for all $t \in [t_0,\infty)$, $q \in Q$.

The remaining $|I \backslash Q| = m - 1$ pursuers cannot carry out a simultaneous m-tuple capture. The theorem is proved.

Remark. It was shown earlier[2] that, when Condition 1 ($m = 1$) is satisfied, an escapee control exists for which either a simultaneous $(\nu + 1)$ -tuple capture occurs place or no capture occurs. In this paper, we consider the problem of a guaranteed simultaneous m-tuple capture, regardless of the actions of the evader.

3. Examples

1° Consider a game Γ_1 of six persons in R^2: the pursuers $P_1,\ldots P_5$ and the evader E, of the form of (1.1) where

$$X_i^0 = \left\| \begin{matrix} \cos\frac{2\pi i}{5} \\ \sin\frac{2\pi i}{5} \end{matrix} \right\|, \quad i = 1, \ldots, 5, \quad Y^0 = \left\| \begin{matrix} 0 \\ 0 \end{matrix} \right\|$$

Note that the initial positions of the pursuers form a regular pentagon with centre at the initial position of the evader. On checking, we find that, when $m \leq 2$, Condition 1 is satisfied and, when $m \geq 3$, it is not satisfied.

Assertion 1. A simultaneous twofold capture is possible in the game Γ_1 and a capture of a greater multiplicity is impossible.

2° We now consider a game Γ_2 of eight persons in R^2: the pursuers $P_1,\ldots P_7$ and the evader E, of the form of (1.1), where

$$X_i^0 = \left\| \begin{matrix} \cos\frac{2\pi i}{7} \\ \sin\frac{2\pi i}{7} \end{matrix} \right\|, \quad i = 1, \ldots, 7, \quad Y^0 = \left\| \begin{matrix} 0 \\ 0 \end{matrix} \right\|$$

Assertion 2. A simultaneous triple capture is possible in the game Γ_2 and a capture of higher multiplicity is impossible.

Generalizing the results of games Γ_1 and Γ_2, we consider the following example.

3° Consider a game Γ_3 of $2+2m$ ($m \geq 1$) persons in R^2: the pursuers $P_1, \ldots P_{1+2m}$ and the evader E, of the form of (1.1), where

$$X_i^0 = \left\| \begin{matrix} \cos\frac{2\pi i}{1+2m} \\ \sin\frac{2\pi i}{1+2m} \end{matrix} \right\|, \quad i = 1, \ldots, 1+2m, \quad Y^0 = \left\| \begin{matrix} 0 \\ 0 \end{matrix} \right\|$$

Assertion 3. A simultaneous m-tuple capture is possible in the game Γ_3 and a capture of higher multiplicity is impossible.

4° Consider a game Γ_4 of $2+3m$ persons ($m \geq 1$) in R^3.: the pursuers $P_1, \ldots P_{1+3m}$ and the evader E, of the form of (1.1), where

$$X_i^0 = \left\| \begin{matrix} \cos\frac{2\pi i}{1+2m} \\ \sin\frac{2\pi i}{1+2m} \\ 0 \end{matrix} \right\|, \quad X_k^0 = \left\| \begin{matrix} 0 \\ 0 \\ 2 \end{matrix} \right\|, \quad Y^0 = \left\| \begin{matrix} 0 \\ 0 \\ 1 \end{matrix} \right\|$$

$$i = 1, \ldots, 1+2m, \quad k = 2+2m, \ldots, 1+3m$$

Assertion 4. A simultaneous m-tuple capture is possible in the game Γ_4 and a capture of a higher multiplicity is impossible.

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